Math 53, Discussions 116 and 118 **2022-09-16**

Lines

Answers included

Questions

Question 1. See the back of this sheet.

Question 2. Find parametric equations for the line contained in the plane $x + y + z = 20$ which also intersects the line $x = y = 2z$ at a right angle.

Question 3. Let L_1 be the line passing through $A(1, -2, 4)$ and $B(2, 1, 3)$, and let L_2 be the line passing through $C(0, 3, -3)$ and $D(2, 4, 1)$.

Are L_1, L_2 parallel, skew, or intersecting? If they intersect, where do they intersect? If not, how far apart are they?

The following are solutions to the problem

"Find the distance d between the point $P(1,-2,2)$ and the line $\mathbf{r}(t) = \langle 3+3t, 2-t, 5t\rangle$."

Figure out what is happening in each one.

(1) Solution 1:

$$\begin{aligned} D^2 &= (2+3t)^2 + (4-t)^2 + (5t-2)^2 \\ &= 35t^2 - 16t + 24 \\ \frac{\mathrm{d}}{\mathrm{d}t}(D^2) &= 70t - 16 = 0 \\ t &= 8/35 \\ d = D_{\min} &= \sqrt{35(8/35)^2 - 16(8/35) + 24} = 2\sqrt{194/35}. \end{aligned}$$

(2) Solution 2:

$$\begin{aligned} 3(x-1) - (y+2) + 5(z-2) &= 0 \\ 3x - y + 5z - 15 &= 0 \\ 3(3+3t) - (2-t) + 5(5t) - 15 &= 0 \\ t &= 8/35 \\ d &= \sqrt{(3+3(8/35)-1)^2 + (2-(8/35)+2)^2 + (5(8/35)-2)^2} = 2\sqrt{194/35}. \end{aligned}$$

(3) Solution 3:

$$\begin{aligned} \langle 1,-2,2\rangle - \langle 3,2,0\rangle &= \langle -2,-4,2\rangle \\ \langle -2,-4,2\rangle \times \langle 3,-1,5\rangle &= \langle -18,16,14\rangle \\ |\langle -18,16,14\rangle| &= 2\sqrt{194} \\ |\langle 3,-1,5\rangle| &= \sqrt{35} \\ d &= 2\sqrt{194/35}. \end{aligned}$$

(4) Solution 4:

$$\begin{aligned} \langle 1,-2,2\rangle - \langle 3,2,0\rangle &= \langle -2,-4,2\rangle \\ (\langle 3,-1,5\rangle \times \langle -2,-4,2\rangle) \times \langle 3,-1,5\rangle &= \langle 94,132,-30\rangle = 2\langle 47,66,-15\rangle \\ \frac{\langle 47,66,-15\rangle \cdot \langle -2,-4,2\rangle}{|\langle 47,66,-15\rangle|} &= -388/\sqrt{6790} \\ d &= |-388/\sqrt{6790}| = 2\sqrt{194/35}. \end{aligned}$$

(5) Solution 5:

$$\begin{aligned} \langle 2+3t, 4-t, 5t-2\rangle \cdot \langle 3,-1,5\rangle &= 0 \\ 35t - 8 &= 0 \\ t &= 8/35 \\ d &= \sqrt{(3+3(8/35)-1)^2 + (2-(8/35)+2)^2 + (5(8/35)-2)^2} = 2\sqrt{194/35} \end{aligned}$$

(6) Solution 6:

$$\begin{aligned} \langle 1,-2,2\rangle - \langle 3,2,0\rangle &= \langle -2,-4,2\rangle \\ \frac{\langle 3,-1,5\rangle \cdot \langle -2,-4,2\rangle}{\langle 3,-1,5\rangle \cdot \langle 3,-1,5\rangle}\langle 3,-1,5\rangle &= \frac{8}{35}\langle 3,-1,5\rangle \\ \langle -2,-4,2\rangle - \frac{8}{35}\langle 3,-1,5\rangle &= \langle -\frac{94}{35}, -\frac{132}{35}, \frac{6}{7}\rangle \\ |\langle -\frac{94}{35}, -\frac{132}{35}, \frac{6}{7}\rangle| &= 2\sqrt{194/35}. \end{aligned}$$

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. We went over all of the solutions in class.

Question 2. We need to find a point on the line as well as a direction vector for the line. For the former, we note that the point of intersection of $x + y + z = 20$ with $x = y = 2z$ will suffice. This point is $(x, y, z) = (8, 8, 4)$.

For the latter, the direction vector must be orthogonal to $\langle 1, 1, 1\rangle$ (a normal vector for the plane) as well as $\langle 1, 1, 1/2\rangle$ (a direction vector for the other line). So we can just take the cross product of these two vectors, obtaining $\langle -1/2, 1/2, 0\rangle$. Hence the final equations of the desired line are

$$x = 8 - t/2, \qquad y = 8 + t/2, \qquad z = 4.$$

Question 3. Here are the equations of the two lines:

$$\mathbf{L}_1(t) = \langle 1, -2, 4\rangle + t\langle 1, 3, -1\rangle,$$
$$\mathbf{L}_2(s) = \langle 0, 3, -3\rangle + s\langle 2, 1, 4\rangle.$$

Inspection of their slope vectors shows that they are not parallel, so they are either skew or intersecting.

By attempting to equate their coordinates and trying to solve for s, t, you will find no solutions, meaning that the lines do not intersect. Hence they must be skew. (Alternatively, once you compute the distance between the lines and find that it is nonzero, that also eliminates the case they are intersecting.)

The distance between them is then given by

$$\left|\operatorname{comp}_{\langle 1,3,-1\rangle\times\langle 2,1,4\rangle}(\langle 1, -2, 4\rangle - \langle 0, 3, -3\rangle)\right| = \left|\operatorname{comp}_{\langle 13,-6,-5\rangle}\langle 1, -5, 7\rangle\right| = \boxed{8/\sqrt{230}}.$$

The cross product $\langle 1, 3, -1\rangle \times \langle 2, 1, 4\rangle$ gives the direction perpendicular to both lines, so the projection of any vector from one line to the other onto that cross product yields the desired distance.